STUDY OF THE GENERALIZED POTENTIAL INTEGRAL

BY

JUDSON SANDERSON, JR.

B.S., University of Illinois, 1947
M.S., University of Illinois, 1948

AN ABSTRACT OF A THESIS

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN MATHEMATICS
IN THE GRADUATE COLLEGE OF THE
UNIVERSITY OF ILLINOIS, 1950

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STUDY OF THE GENERALIZED POTENTIAL INTEGRAL

The purpose of the paper is the study of integrals of the form

$$I = \int_{v} \cdots \int d\mu / R(a,\rho),$$

where v is a bounded n dimensional Euclidean space, μ is an additive function of Lebesgue measurable sets, and $R(a,p)$ is the distance between a and the variable point of integration, p.

In the theory of gravitational attraction it is assumed that any two particles of matter are attracted toward each other with a force which varies directly as the product of their masses and inversely as the square of the distance between them. From this assumption is derived the potential $F(a)$ at the point a for a spatial distribution of mass:

$$\text{II} \quad F(a) = \int_{v} \cdots \int \rho(p)\, dp / R(a,p)$$

where $\rho(p)$ is a function of point representing the density of mass at the point $p = (x_1, x_2, \cdots, x_n)$. Since the integral I resembles the potential integral II and actually includes it as a special case, I is called the generalized potential integral.

Among the properties of integrals I studied in the paper are convergence, differentiability, and continuity of I, approximation of I by sequences of harmonic functions, and representation by I of certain classes of functions of n real variables. Many of the results are applied to the related series $\sum_{K} A_K / R(a_K, a)$.

The investigation of integrals I parallel many of the developments of W. J. Trjitzinsky in his paper, "Some General Developments in the Theory of Functions of a Complex Variable," Acta Mathematica, vol. 70, pp. 63-163. In this

paper he studies functions of the form $F(a) = \int\int d\mu/z - a$, where z is the complex variable $x + i\,y$, and where integration is extended over sets in the complex plane.

Just as in Professor Trjitzinsky's paper, it is of importance to consider the notion of symmetric density of an additive function of sets: Let $S(a,r)$ be the closed sphere with center a and radius r. Symmetric density $\rho(a)$ of the additive set function μ at the point a is defined to be:

$$\lim_{r \to o} 1/V(S(a,r)) \int\int_{S(a,r)} - \int |\, d\mu \,| = \lim_{r \to o} \rho(a,r) = \rho(a),$$

where $V(S(a,r))$ is the volume of $S(a,r)$. Let a be a point for which the density $\rho(a)$ of μ exists. The speed with which $\rho(a,r)$ approaches $\rho(a)$ as $r \to o$ is called the rarefication of μ at a. The basic principle which evolves in the study of I is: The greater the rarefication of μ, the more regularity properties I will have.

For simplicity the investigations are confined to three dimensions. Unless otherwise specified, the extension to $n > 3$ dimensions is immediate.

VITA

Judson Sanderson, Jr. was born in Orrick, Missouri, July 15, 1921. He was graduated from Central High School of Kansas City, Missouri, in June 1938. He attended Kansas City Junior College from September 1940 to June 1943, graduating A.A. with honors in 1942 and A.S. with honors in 1943. In June 1943 he was called to active duty in the United States Army and served 32 months, 16 months of which in the China-Burma-India and Pacific theaters of operations, and 9 months of which in the A.S.T.P. program at the University of Illinois and at V.M.I. He returned to the University of Illinois in June of 1946 and graduated B.S. with highest honors in mathematics February 1947. He thereupon entered the Graduate College and became at the same time a half-time teaching assistant in mathematics. In June 1948 he received the M.S. degree. During the academic years 1948-1949 and 1949- 1950 he has been a Fellow in Mathematics at the University of Illinois with full time spent in graduate study.